Partitioning $\mathbb{R}^3$ in unit circles

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Introduction

The study of *paradoxical sets* has been of interest to mathematicians for a long time. Paradoxical sets are subsets of the reals or of $\mathbb{R}^n$, whose existence is proved using the Axiom of Choice. We focus here on one type of paradoxical set: **partitions of $\mathbb{R}^3$ into unit circles (*PUC*)**.

Research objectives

- Understand whether we can recover some fragment of Choice from the existence of one or many paradoxical sets.
- Overcome the geometric-combinatorial obstacles that arise while trying to apply known methods to the particular case of PUCs.

Framework

- In ZF, $\mathbb{R}^3$ can be partitioned into circles [4].
- In ZF**C**, $\mathbb{R}^3$ can be partitioned into *unit* circles [3]. Here, the construction of the PUC is defined recursively using a well-order of $\mathbb{R}$, as Figure 1 shows.

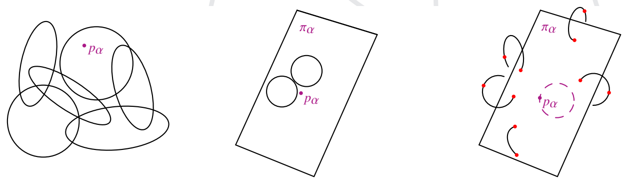

Figure 1: I. In step α define a circle that covers the point p_α. II. Choose a plane π_α that does *not* contain previous circles. III. By cardinality, pick a circle contained in π_α that avoids all the red points.

We prove that there is a model of ZF with such a partition without a well-order of the reals.

Forcing such partition

We follow the structure of the construction of a model of ZF with a Hamel basis of $\mathbb{R}$ but no well-order of the reals done in [2]. We obtain a model W, where

$$W \models ZF + DC + \text{no well-order of } \mathbb{R} + \text{there is a PUC.}$$

In the Cohen model

It turns out we can already find a PUC in a well known model H: the Cohen–Halpen–Levy model. We adapt the proof in [1], where it is shown that there is a Hamel basis of $\mathbb{R}$ in H. We then get

$$H \models ZF + \neg AC_\omega + \text{there is a PUC.}$$

Important fact: The reals in H can be organized in the following way:

$$\mathbb{R} \cap H = \bigcup_{a \in [A]^{<\omega}} \mathbb{R} \cap L[a],$$

where each piece $\mathbb{R} \cap L[a]$ has a canonical well-ordering inside H. We use this to construct a PUC by taking care of one *"layer"* of reals at a time.

The obstacles that arise

Some geometric issues relating to the partitions in unit circles have to be dealt with. Let a, b and c be mutually generic Cohen reals.
These *obstacles* are represented on Figure 2.

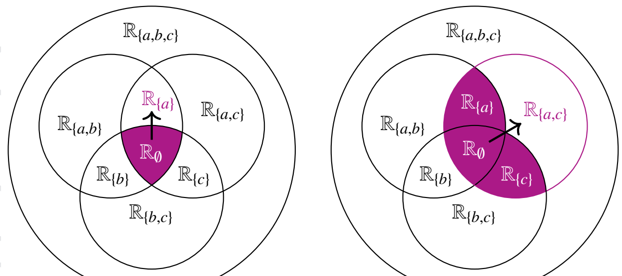

Figure 2: Representation of the obstacles: I. *Extendability* and II. *Amalgamation* (for $n = 2$).

Extendability
Can we extend a PUC of L to $L[a]$?

Amalgamation (n=2)
Can we extend (compatible) PUCs of $L[a]$ and $L[c]$ to a PUC in $L[a, c]$?

Algebraic detour and further questions

In order to overcome the above obstacles, some natural questions about transcendence degrees of set theoretical sets of reals arise.

Fact: Let V be a model of ZFC and let $V[c]$ be a generic extension obtained by adding one Cohen real. Then the transcendence degree of $\mathbb{R}^{V[c]}$ over $\mathbb{R}^V$ is $\mathfrak{c}$.

Work in progress: *Can we extend this result to any finite number of (combined) Cohen reals? And to any type of reals?*

References

[1] Mariam Beriashvili, Ralf Schindler, Liuzhen Wu, and Liang Yu. Hamel bases and well-ordering the continuum.

[2] Jorg Brendle, Fabiana Castiblanco, Ralf Schindler, Liuzhen Wu, and Liang Yu. A model with everything except for a well-ordering of the reals.

[3] J. H. Conway and H. T. Croft. Covering a sphere with congruent great-circle arcs. *Mathematical Proceedings of the Cambridge Philosophical Society*, 60:787–800, 1964.

[4] Andrzej Szulkin. R^3 is the union of disjoint circles, 1983.