Universität Münster

Paradoxical sets
and the Axiom of Choice

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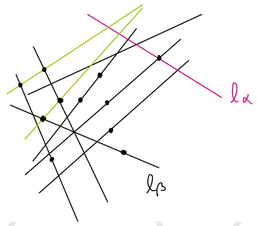
Paradoxical sets are subsets of the reals or of $\mathbb{R}^n$, whose existence is proved using the Axiom of Choice. We focus here on the following paradoxical sets.

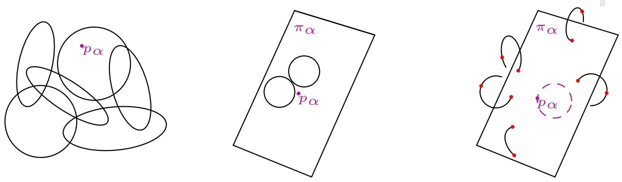
Objects

- **Hamel basis:** A basis of $\mathbb{R}$ as $\mathbb{Q}$-vector space.

- **Mazurkiewicz set:** A subset A of $\mathbb{R}^2$ such that, for every line l on the plane, $|A \cap l| = 2$ [5].



- **PUC:** A partition of $\mathbb{R}^3$ into unit circles [4].



Cooking a model of $\mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WO}(\mathbb{R}) + \psi(\mathcal{P})$

Dish: $L(\mathbb{R}, \cup h)^{V[g,h]}$

Ingredients:

- $\mathbf{Q} = \mathbf{C}(\omega_1)$ be the finite support product of ω_1-many copies of Cohen forcing,
- g be a $\mathbf{Q}$-generic filter over V,
- $\mathbf{P} \in V[g]$: $p \in \mathbf{P} \iff \exists x \in \mathbb{R}\ V[x] \models \psi(p)$[a],
- h be a $\mathbf{P}$-generic filter over $V[g]$.

Preparation:

1. Check $\mathbf{P}$ be a forcing notion over $V[g]$ that *adds a real partition* and is σ-closed.
2. Mix up h for 10 minutes until you get $\mathcal{P} = \cup h$.
3. Prove that $\mathbf{P}$ satisfy *Extendability*.
4. Prove that $\mathbf{P}$ satisfy *Amalgamation*.
5. Salt to taste.

[a] $\psi(p) : p \subseteq \mathbb{R}^n, \forall s \in [p]^{<\omega} \psi_1(s) \wedge \forall r \in \mathbb{R}^m \exists s \in [p]^{<\omega} \psi_2(r, s)$

Research objectives

- Understand whether we can recover some fragment of Choice from the existence of one or many paradoxical sets.
- Overcome the geometric-combinatorial obstacles that arise in each paradoxical case

Results

	Hamel basis	Mazurkiewicz	Partition of $\mathbb{R}^3$ in unit circles
$\mathsf{DC} + \neg\mathsf{WO}(\mathbb{R})$	$L(\mathbb{R},h)^{L[\tilde{g},h]}$ [3*]	$L(\mathbb{R},h)^{L[\tilde{g},h]}$ [3*]	$L(\mathbb{R},h)^{L[g,h]}$
$\mathsf{DC} + \neg\mathsf{Uf}(\omega)$	$L(\mathbb{R},h)^{L[g,h]}$	$L(\mathbb{R},h)^{L[g,h]}$?	?
$\neg\mathsf{AC}_\omega$	$\mathrm{HOD}^{L[g]}_{A\cup\{A\}}$ [2]	$\mathrm{HOD}^{L[g]}_{A\cup\{A\}}$ [1]	$\mathrm{HOD}^{L[g]}_{A\cup\{A\}}$

Algebraic detour

Proving that the forcing that adds the paradoxical set we want satisfies Extendability and Amalgamation involves specific properties of the specific set. For example, the case of PUCs, we need algebraic help. Some natural questions about transcendence degrees of set theoretical sets of reals arise.

Fact: Let V be a model of ZFC and let $V[c]$ be a generic extension obtained by adding one Cohen real. Then the transcendence degree of $\mathbb{R}^{V[c]}$ over $\mathbb{R}^V$ is $\mathfrak{c}$.

Theorem. *Let X be a finite set of mutually generic Cohen reals over V. In $V[X]$, consider the minimum field $F \subseteq \mathbb{R}$ such that $F \supseteq \bigcup_{Y \subsetneq X} \mathbb{R}^{V[Y]}$. Then, in $V[X]$ the transcendence degree of $\mathbb{R}$ with respect to F is continuum.*

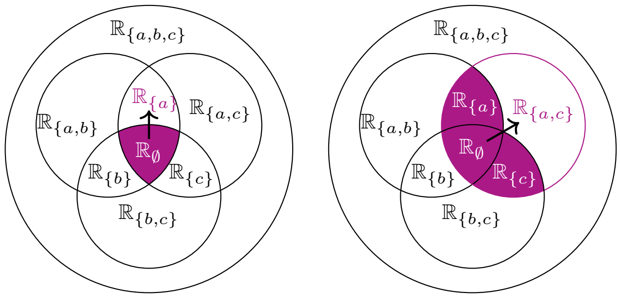


Figure 1: a, b and c are mutually generic Cohen reals. I. *Extendability* and II. *Amalgamation* (for $n = 2$).

References

[1] Mariam Beriashvili and Ralf Schindler. Mazurkiewicz sets with no well-ordering of the reals. *Georgian Mathematical Journal*, 29, 03 2022.

[2] Mariam Beriashvili, Ralf Schindler, Liuzhen Wu, and Liang Yu. Hamel bases and well-ordering the continuum. *Proceedings of the American Mathematical Society*, 146:1, 03 2018.

[3] Jörg Brendle, Fabiana Castiblanco, Ralf Schindler, Liuzhen Wu, and Liang Yu. A model with everything except for a well-ordering of the reals. *arXiv:1809.10420*, 2018.

[4] J. H. Conway and H. T. Croft. Covering a sphere with congruent great-circle arcs. *Mathematical Proceedings of the Cambridge Philosophical Society*, 60:787 – 800, 10 1964.

[5] S. Mazurkiewicz. Sur un ensemble plan qui a avec chaque droite deux et seulement deux points communs. *C.R. Soc. de Varsovie 7*, pages 382–383, 1914.

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